Waldspurger Matrices

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In 2005 J. L. Waldspurger proved the following theorem: Given a finite real reflection group G, the closed positive root cone is tiled by the images of the open weight cone under the action of the linear transformations $1-g$. Shortly after this E. Meinrencken extended the result to affine Weyl groups and then P. V. Bibikov and V. S. Zhgoon gave a uniform proof for a discrete reflection group acting on a simply-connected space of constant curvature.

In this paper we show that the Waldspurger and Meinrencken theorems of type A give an interesting new perspective on the combinatorics of the symmetric group. In particular, for each permutation matrix $g \in \mathfrak{S}_n$ we define a non-negative integer matrix $\mathbf{WT}(g)$, called the *Waldspurger transform* of g. The definition of the matrix $\mathbf{WT}(g)$ is purely combinatorial but it turns out that its columns are the images of the fundamental weights under $1-g$, expressed in simple root coordinates. The possible columns of $\mathbf{WT}(g)$ (which we call *UM vectors*) biject to many interesting structures including: unimodal Motzkin paths, abelian ideals in the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, Young diagrams with maximum hook length n, and integer points inside a certain polytope.

We show that the sum of the entries of $\mathbf{WT}(g)$ is half the entropy of the corresponding permutation g, which is known to equal the rank of g in the MacNeille completion of the Bruhat order. Inspired by this, we extend the Waldpurger transform $\mathbf{WT}(M)$ to alternating sign matrices M and give an intrinsic characterization of the image. This provides a geometric realization of MacNeille completion of the Bruhat order (a.k.a. the poset of alternating sign matrices).