Trees for Given Values of the Span, Caps and Icaps for $L(2,1)$-Colorings

V. Coufal$^2$, K. Fogel$^1$, A. Higgins$^3$, W. Higgins$^4$, R. Ray$^2$, J. Villalpando$^*$ $^1$, K. Yerion$^2$

$^1$ California Lutheran University, $^2$ Gonzaga University, $^3$ University of Dayton, $^4$ Wittenberg University

An $L(2,1)$-coloring of a graph is a labeling of the vertices using non-negative integers such that adjacent vertices differ in label by at least 2 and distance two vertices differ in label. The span of an $L(2,1)$-coloring is the smallest integer $\lambda$ for a given graph such that there exists an $L(2,1)$-coloring of the graph using only non-negative integers less than or equal to $\lambda$. The invariant caps, denoted $\overline{\kappa}$, is the least number of color classes required to create an $L(2,1)$-coloring on a given graph. An $L(2,1)$-coloring of a graph is irreducible if reducing the label on any vertex violates an $L(2,1)$-coloring condition. The invariant icaps, denoted $\kappa$, is the least number of color classes required to create an irreducible $L(2,1)$-coloring on a given graph. For any tree $T$ it is known that $\Delta + 1 \leq \overline{\kappa} \leq \kappa \leq \lambda + 1$ and $\lambda \in \{\Delta + 1, \Delta + 2\}$ where $\Delta$ is the maximum degree of the tree. Thus, there are only three possible values for $\overline{\kappa}$ and $\kappa$: $\Delta + 1, \Delta + 2, \Delta + 3$. We prove that $\overline{\kappa} = \Delta + 1$ for all trees. Then for each of the two possible values of $\lambda$, we consider the three possible values of $\kappa$, determine if there exists a tree with the two specified values of $\lambda$ and $\kappa$, and provide a family of such trees if any exist.

Keywords: $L(2,1)$-coloring, irreducible $L(2,1)$-coloring, caps, icaps, span